※ Foundations 2a: Introduction to Qubits (1/10)

1.1 A Single Qubit

We are now ready to begin our discussion of quantum states. Let's start by defining a quantum bit or qubit which is a quantum state that can model a binary system.

Definition 1.1. A **qubit** is an object which can be represented using a *unit vector* with complex **amplitudes** α_0 and α_1 as

$$|\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}, \tag{1}$$

where we say that α_i is the amplitude corresponding to the event i for $i \in \{0, 1\}$. The notation $|\psi\rangle$ is read as "ket" "psi".

A unit vector is a vector with length 1. The length of a vector can be found by calculating

$$|| \, |\psi\rangle \, || = \sqrt{|\alpha_0|^2 + |\alpha_1|^2} \tag{2}$$

Question 1. Which of the following represents a qubit?

$$|\psi_1\rangle = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \quad \cancel{|\psi_2\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}} \quad |\psi_3\rangle = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tag{3}$$

$$|| \, |\psi_1\rangle \, || = \sqrt{\left|\tfrac{1}{\sqrt{3}}\right|^2 + \left|\sqrt{\tfrac{2}{3}}\right|^2} = \sqrt{\tfrac{1}{3} + \tfrac{2}{3}} = 1.$$

$$|| \, |\psi_2\rangle \, || = \sqrt{\left|\tfrac{1}{2}\right|^2 + \left|\tfrac{1}{2}\right|^2} = \sqrt{\tfrac{1}{2}} \neq 1$$

$$|| \, |\psi_3\rangle \, || = \sqrt{|\cos\theta|^2 + |\sin\theta|^2} = \sqrt{\underbrace{\cos^2\theta + \sin^2\theta}_{1}} = \sqrt{1} = 1$$

$$\left|\left| \begin{bmatrix} i/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right|\right| = \sqrt{\left|\tfrac{i}{\sqrt{2}}\right|^2 + \left|-\tfrac{1}{\sqrt{2}}\right|^2} = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \tfrac{1}{2}} = 1 \quad \checkmark$$

$$\hookrightarrow \left|\tfrac{i}{\sqrt{2}}\right| = \sqrt{a^2 + b^2} = \sqrt{0 + \left(\tfrac{1}{\sqrt{2}}\right)^2} = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2} = \tfrac{1}{\sqrt{2}}$$

Two of the most important states that we will be using throughout the course are the **standard basis states**, defined as

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4}$$

These are the vectors corresponding to the two primary states that the system can be in. We can use linearity of vectors to write equation (1) as a linear combination of the standard basis states:

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a|0\rangle + b|1\rangle$$

When both coefficients are nonzero, we say that the state $|\psi\rangle$ is in **superposition**.

amplitude.

We are now equipped with the language to represent the state of a qubit. How do we interpret this state? A crucial operation in quantum computing is measurement, which, for our purposes is the way we read out the result of a quantum algorithm. If we measure $|\psi\rangle$ in the standard basis,

$$|\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

- with probability $|\alpha_0|^2$: we *observe* the outcome $|0\rangle$, and the qubit *collapses* to $|0\rangle$.
- with probability $|\alpha_1|^2$: we *observe* the outcome $|1\rangle$, and the qubit *collapses* to $|1\rangle$.

Similar to the case of the probability vector, *observation* collapses the state to the one that we observe. The key difference that makes quantum states special is the fact that there are physical particles which can truly represent superposition, whereas our discussion around probability vectors was slightly superficial.

Question 2. Suppose we have the state

$$|\phi\rangle = \overbrace{\left(\frac{1}{\sqrt{6}} - i\frac{1}{\sqrt{6}}\right)}^{\alpha_0}|0\rangle + \left(\frac{1}{\sqrt{3}} + i\frac{1}{\sqrt{3}}\right)|1\rangle\,. \tag{5}$$

if we obs. $|0\rangle$.

What is the probability of measuring $|0\rangle$, and what is the state after the measurement?

$$|\alpha_0|^2 = \alpha_0^* \alpha_0 = \left(\frac{1}{\sqrt{6}} + i\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}} - i\frac{1}{\sqrt{6}}\right) = \left(\frac{1}{6} + \frac{1}{6}\right) = \frac{1}{3}$$

After measurement: $|0\rangle$.

1.2 Multiple Qubits Sneak Peek

To represent a probability distribution over more than two states, we simply use a probability vector with the number of states we need.

Question 3. Suppose we had n bits available. How many states can we represent?

2^n states.

We can extend our system to multiple qubits the same way we did for probability vectors. Instead of probabilities for events occurring, each event x has an associated complex number α_x called its **amplitude**. As a vector, this would look like

$$|\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{2^n-1} \end{bmatrix}. \tag{6}$$

$|\alpha_0|^2 + |\alpha_1|^2 + \cdots + |\alpha_{2^n-1}|^2 = 1.$

We use the tensor product notation again to combine systems of states. For example, a system of two qubits in the $|0\rangle$ would be written as

$$\begin{bmatrix} 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

Again, we require that the vector is a unit vector: $\sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1$. This ensures that the squared norm of the amplitudes form a probability distribution.

Question 4. Write down a 2-qubit state where the probability of measuring each qubit is equal.

$$\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \qquad \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad \begin{bmatrix} i/2 \\ -i/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

prob. vec

2 qubits

$1/2 \to |01\rangle \quad 1/4$

$$\left|-\tfrac{i}{2}\right|^2 = \left(\sqrt{\tfrac{i}{2} \times \left(-\tfrac{i}{2}\right)}\right)^2 = \frac{1}{4}$$

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

1.3 Transformations

Quantum algorithms have three main components:

1. Store quantum information (statevector)
2. Manipulate quantum information (unitary transformations)
3. Extract some output (quantum measurement)

We've seen examples of what 1 and 3 look like, so here we will briefly discuss 2. The manipulation of quantum information can be thought of as a transformation from one quantum state to a new quantum state, which can be expressed in vector form as

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \xrightarrow{U} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{N-1} \end{bmatrix}. \tag{8}$$

One requirement we have for these transformations is that they be linear. This means that if we know what a transformation does for all basis vectors, we will know how *any* vector will be transformed. We will review this more carefully in the next section, so here we explore an example for single qubit states.

Question 5. Suppose we have a linear transformation T that acts as follows on the standard basis states:

$$T|0\rangle = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \qquad T|1\rangle = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \tag{9}$$

What is the action of T on the state $|\psi\rangle := \gamma_0|0\rangle + \gamma_1|1\rangle$?

$$T|\psi\rangle = T\left(\gamma_0|0\rangle + \gamma_1|1\rangle\right)$$

$$= \gamma_0 T|0\rangle + \gamma_1 T_1|1\rangle$$

$$= \gamma_0 \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} + \gamma_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} \gamma_0 a_0 + \gamma_1 a_1 \\ \gamma_0 b_0 + \gamma_1 b_1 \end{bmatrix}$$

※ Foundations 3a: Linear Algebra (1/12)

Hilbert space is a big place.

- Carlton Caves

2.1 Vector Spaces

In linear algebra, we are interested in studying vector spaces.

Definition 2.1 (Vector Space)**.** A **vector space** is a set of elements that is closed under linear combinations. A **linear combination** is a combination of vectors via vector addition and scalar multiplication.

The primary focus of this course will be the **complex vector space** of N dimensions, which will be referred to via the short hand $\mathbb{C}^N$. You may also see me (and others) refer to the "Hilbert space" of quantum states. These are the same thing, as a Hilbert space can be thought of as a vector space where you can take inner products. Elements of $\mathbb{C}^N$ are vectors of the form

$$|v\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}. \tag{10}$$

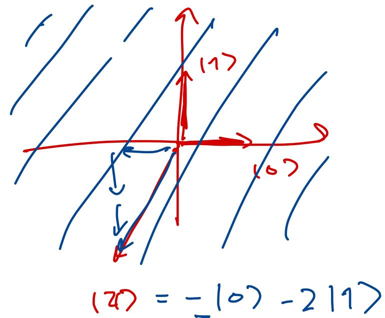
You may be familiar with using $\vec{v}$ to represent vectors, but here we will use $|v\rangle$ to represent column vectors.

Question 6. Verify that $\mathbb{C}^N$ is indeed a vector space. I.e., are the elements closed under linear combinations.

2.2 Span and Linear Independence

Definition 2.2 (Span)**.** The **span** of a set of N vectors $\{|\psi_1\rangle, \ldots, |\psi_N\rangle\}$ is the set of all linear combinations of $|\psi_1\rangle, \ldots, |\psi_N\rangle$, i.e. the set of all states that can be written as

$$c_1 |\psi_1\rangle + \cdots + c_N |\psi_N\rangle \tag{11}$$



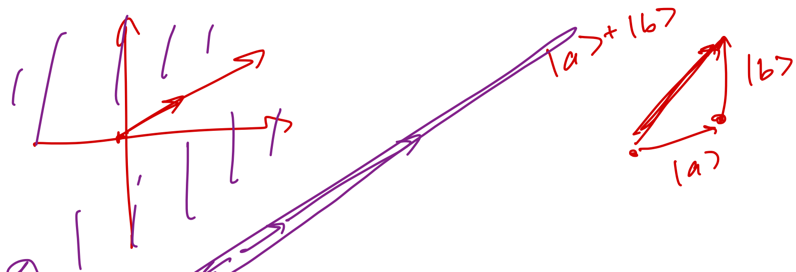
for all complex scalars $c_1, \ldots, c_N \in \mathbb{C}$.

span $\{|0\rangle, |1\rangle\} \equiv c_0|0\rangle + c_1|1\rangle$

$\mathbb{C}^2 \leftarrow$ Space of all 2-dim vecs w/ complex entries.

Question 7. If the following statement is true, prove it. If not, provide a counterexample:

For any pair of length two vectors with real entries $|v\rangle, |w\rangle$, the span of $|v\rangle$ and $|w\rangle$ is all of $\mathbb{R}^2$. In other words, any two pair of vectors spans the entire space.



Definition 2.3 (Linearly Independent Set of Vectors)**.** Let $B = \{|\psi_1\rangle, \ldots, |\psi_N\rangle\}$ be a set of vectors in $\mathbb{C}^N$. We say that this set of vectors is **linearly independent** if

$$c_1 |\psi_1\rangle + \cdots + c_N |\psi_N\rangle = 0 \tag{12}$$

if and only if $c_i = 0$ for all i.

The above definition is equivalent to saying that no basis vector can be written as a linear combination of the other basis vectors. Equivalently, we say that a set of vectors is independent if for any $|\phi\rangle \in \mathbb{C}^N$, there is a *unique* set of scalars $c_1, \ldots, c_N \in \mathbb{C}^N$ such that

$$c_1 |\psi_1\rangle + \cdots + c_N |\psi_N\rangle = |\phi\rangle\,. \tag{13}$$

dot product.

2.3 Inner Products and Bases

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 + 12 = 14$

We can equip a vector space with an inner product, which is an operation that maps two vectors to a scalar value. We will refer to a vector space with an inner product a **Hilbert space**.

Definition 2.4 (Inner Product ($\mathbb{C}^N$)). Let $|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}$ and $|\phi\rangle = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix}$ be elements of $\mathbb{C}^N$. Then the **inner product** between $|\psi\rangle$ and $|\phi\rangle$ is $\sum_i \alpha_i^* \beta_i$. In matrix product form, an equivalent way to write this is

$$\begin{bmatrix} \alpha_1^* & \cdots & \alpha_N^* \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} = \sum_i \alpha_i^* \beta_i. \tag{14}$$

In ket notation, we write the **dual** of a complex vector $|\psi\rangle$ as $\langle\psi| := \begin{bmatrix} \alpha_1^* & \cdots & \alpha_N^* \end{bmatrix}$, read as **bra** psi. Using this notation, the inner product is often written as $\langle\psi|\phi\rangle$. We refer to the notation of writing vectors with these angle brackets as **bra-ket notation**.

Question 8. Let $|\phi\rangle = \begin{bmatrix} i/\sqrt{3} \\ \sqrt{\frac{2}{3}} \end{bmatrix}$ and $|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix}$. Calculate $\langle\psi|\phi\rangle$ and $\langle\phi|\psi\rangle$.

$\langle\psi|\phi\rangle = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix} \begin{bmatrix} i/\sqrt{3} \\ \sqrt{2/3} \end{bmatrix} = i/\sqrt{6} + i/\sqrt{3}$

$\langle\phi|\psi\rangle = \begin{bmatrix} -i/\sqrt{3} & \sqrt{2/3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix} = -i/\sqrt{6} - i/\sqrt{3}$

$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$

Question 9. What happens when we take the inner product of a vector with itself? Does it relate to a quantity about vectors you've seen before?

$|\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$

$\langle\psi|\psi\rangle = \begin{bmatrix} \alpha_0^* & \alpha_1^* \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \alpha_0^*\alpha_0 + \alpha_1^*\alpha_1 = |\alpha_0|^2 + |\alpha_1|^2$

$\| |\psi\rangle \|^2$

Definition 2.5 (L2-norm)**.** For a vector $|\psi\rangle \in \mathbb{C}^N$, the **L2-norm** of $|\psi\rangle$ denoted is

$$|| \, |\psi\rangle \, || := \sqrt{\langle\psi|\psi\rangle}. \tag{15}$$

In this course, when we say norm, we will be referring to the L2-norm of the vector unless otherwise stated. If the norm of a vector is 1, we say that vector is a **unit vector**.

Question 10. What is the norm of $|\psi\rangle = (2+i)\,|0\rangle + (3-2i)\,|1\rangle$?

$$|\psi\rangle = \begin{bmatrix} 2+i \\ 3-2i \end{bmatrix}$$

$$|| \, |\psi\rangle \, || = \sqrt{\langle\psi|\psi\rangle} = \sqrt{\begin{bmatrix} 2-i & 3+2i \end{bmatrix}\begin{bmatrix} 2+i \\ 3-2i \end{bmatrix}} = \sqrt{4+1+9+4} = \sqrt{18}$$

$$|\phi\rangle = \frac{1}{\sqrt{18}}\,|\psi\rangle$$

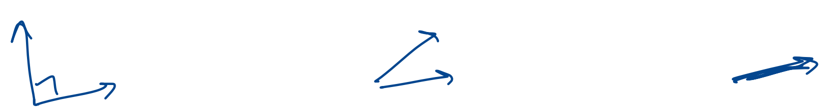
$$\langle\phi|\phi\rangle = \langle\psi|\frac{1}{\sqrt{18}}\cdot\frac{1}{\sqrt{18}}|\psi\rangle = \frac{1}{18}\langle\psi|\psi\rangle = \frac{18}{18} = 1.$$

Dividing a vector by its norm gives a unit vector.
normalization

Definition 2.6 (Orthogonality)**.** Given two vectors $|\psi\rangle$ and $|\phi\rangle$ in $\mathbb{C}^N$, we say that they are **orthogonal** if $\langle\psi|\phi\rangle = 0$.

The inner product is a useful metric in defining a notion of similarity between two vectors. We roughly say a high inner product between two vectors means they have high overlap, and they point in similar directions.

For unit vectors.



$\langle\psi|\phi\rangle = 0$ . . . . $\langle\psi|\phi\rangle = 1$.

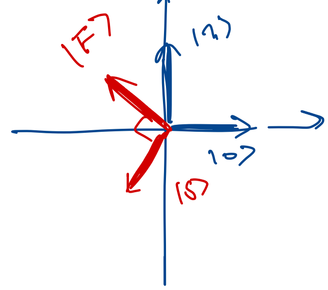
Definition 2.7 (Orthogonal Basis)**.** If a set of N vectors $B = |\psi_1\rangle, \ldots, |\psi_N\rangle$ in $\mathbb{C}^N$ is mutually orthogonal (i.e., if $i \neq j$ then $\langle\psi_i|\psi_j\rangle = 0$), we say that B forms an **orthogonal basis** for $\mathbb{C}^N$.

Furthermore, if every vector $|\psi_i\rangle$ is also a unit vector, we call it an **orthonormal basis**.

$|0\rangle, |1\rangle$.

Question 11. Write a set of orthonormal basis vectors for $\mathbb{R}^2$ besides the standard basis and draw it on the plane. Find an orthonormal basis for $\mathbb{C}^4$ where one of the vectors is $|\psi\rangle \rightarrow \boxed{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}$

$\{|S\rangle, |F\rangle\}.$



$$|\psi\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \qquad [a \;\; b \;\; c \;\; d] \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = 0 = \frac{a}{\sqrt{2}} + \frac{d}{\sqrt{2}} \Rightarrow a = -d$$

$$|\psi_2\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \text{Normalize to be } \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{bmatrix}.$$

Two more are $\left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$

2.4 Summary

We have now covered the main foundational mathematical concepts we will be using to build our understanding of quantum computing. One thing I have really enjoyed about quantum computing is that it gave me a new way to visualize and understand the above tools, which you may have felt were quite abstract in your preliminary courses. I hope this new angle will give you a new appreciation and understanding of these tools. Next week we will start looking at small quantum systems and get familiar with the circuits we will use to prove ideas about the limits of information and construct algorithms.